

Laplace transform associated with the Weierstrass transform

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Abstract:

An elegant expression is obtained for the Laplace-Weierstrass transform LW of the function in terms of their existence. It is also shown , how the main result can be extended to hold for the LW transform of several functions.

Keywords: Transform,Laplace Transform,Weierstrass Transform, Laplace-Weierstrass transform, Generalized Function, Existence theorem,Multiplication Theorem.

1. Introduction:

In this paper we have introduced the concept of LW transform which has use in several fields. The basic idea behind any transform is that the given problem can be solved more readily in the transform domain. The method is especially attractive in the linear mathematical models for physical systems such as a spring/mass system or a series electrical circuit which involve discontinuous functions.

The distributional Laplace transform is defined as

$$F(s) = \langle f(t), e^{-st} \rangle \quad (1.1)$$

The conventional Weierstrass transform is defined by

$$F(x) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} f(y) e^{-(x-y)^2/4} dy \quad (1.2)$$

where $f(y)$ is a suitably restricted conventional function on $-\infty < y < \infty$ and x is a complex variable.

Our purpose in this work is to define and study the Laplace transform associated with the Weierstrass transform.

This transform is defined by

$$F(s, x) = LW\{f(t, y)\} = \frac{1}{\sqrt{4\pi}} \int_0^{\infty} \int_0^{\infty} f(t, y) e^{-st - \frac{(x-y)^2}{4}} dy dt \quad (1.3)$$

2. The Testing Function Space $LW_{a,b}$:

Let a and b be fixed numbers in R^1 , let $f(t, y)$ variable in R^1 , and let $h_{a,b}(t, y)$ denote the function:

$$h_{a,b}(t, y) = \begin{cases} e^{\frac{-ay}{2}}, & -\infty < y < 0 \\ e^{\frac{-by}{2}}, & 0 \leq y < \infty \end{cases} \quad (2.1)$$

$LW_{a,b}$ as the linear space of all complex valued smooth functions $\phi(t, y)$ on $0 < t < \infty$,

$-\infty < y < \infty$ such that for each $p, q = 0, 1, 2, \dots$

$$\gamma_{a,b,p,q} \phi(t, y) = \sup_{\substack{0 < t < \infty \\ 0 < y < \infty}} \left| e^{at + \frac{y^2}{4}} h_{a,b}(t, y) D^{p+q} \phi(t, y) \right| < \infty \quad (2.2)$$

The space $LW_{a,b}$ is complete and a Frechet space. This topology is generated by the total families of countably multinorms space given by (2.2).

3. Existence Theorem for LW Transform:

If $f(t, y)$ is a function which is piecewise continuous on every finite interval in the range $(t, y) \geq 0$ and satisfies $|f(t, y)| \leq M' e^{at} e^{\frac{y^2}{4t_0}}$. For all $(t, y) \geq 0$ and for some constant a, t_0 and M' , then the LW transform of $f(t, y)$ exists for all $s > a$ and $x > 0$.

Proof: It is given that the function $f(t, y)$ is piecewise continuous on every finite interval in the range $(t, y) \geq 0$.

$\Rightarrow f$ is R -integrable over any finite interval in the range $(t, y) \geq 0$.

$\Rightarrow e^{-st - \frac{(x-y)^2}{4}} f(t, y)$ is R -integrable over any finite interval in the range $(t, y) \geq 0$.

Now by definition of LW transform,

$$LW \{f(t, y)\} = \frac{1}{\sqrt{4\pi}} \int_0^\infty \int_0^\infty f(t, y) e^{-st - \frac{(x-y)^2}{4}} dy dt \quad (3.1)$$

$$\leq \frac{1}{\sqrt{\pi}} \int_0^\infty \int_0^\infty M' e^{at} e^{\frac{y^2}{4t_0}} e^{-st - \frac{(x-y)^2}{4}} dy dt$$

$$\leq \frac{-M' e^{-\frac{x^2}{4}}}{(s-a)x\sqrt{\pi}}$$

$$\leq \frac{M e^{-\frac{x^2}{4}}}{x\sqrt{\pi}(s-a)}$$

where- $M' = M$

But $\frac{M e^{-\frac{x^2}{4}}}{x\sqrt{\pi}(s-a)}$ is finite quantity.

$\therefore LW \{f(t, y)\}$ exists provided, $s > a$ and $x > 0$.

4. Theorem: If $LW \{f(t, y)\} = F(s, x)$ then

$$i) LW \{t^n f(t, y)\} = (-1)^n \frac{d^n}{ds^n} F(s, x)$$

and

$$ii) LW \{(x-y)^n f(t, y)\} = (-1)^n (2)^n \frac{d^n}{dx^n} F(s, x)$$

Proof: Given that, $LW \{f(t, y)\} = F(s, x)$

i) By definition of LW transform

$$F(s, x) = LW \{ f(t, y) \} \\ = \frac{1}{\sqrt{4\pi}} \int_0^\infty \int_0^\infty f(t, y) e^{-st - \frac{(x-y)^2}{4}} dy dt \quad (4.1)$$

$$F'(s, x) = \frac{d}{ds} \{ F(s, x) \} \\ = \frac{d}{ds} \left\{ \frac{1}{\sqrt{4\pi}} \int_0^\infty \int_0^\infty f(t, y) e^{-st - \frac{(x-y)^2}{4}} dy dt \right\} \quad (4.2)$$

$$= \frac{-1}{\sqrt{4\pi}} \int_0^\infty \int_0^\infty t f(t, y) e^{-st - \frac{(x-y)^2}{4}} dy dt$$

$$(-1)F'(s, x) = LW \{ t f(t, y) \}$$

or

$$LW \{ t f(t, y) \} = (-1) \frac{d}{ds} F(s, x) \quad (4.3)$$

i.e. the theorem is true for $n = 1$.

Next, let this theorem be true for $n = r$, then we have

$$LW \{ t^r f(t, y) \} = (-1)^r \frac{d^r}{ds^r} [F(s, x)]$$

$$(-1)^r F^{(r)}(s, x) = LW \{ t^r f(t, y) \}$$

$$= \frac{1}{\sqrt{4\pi}} \int_0^\infty \int_0^\infty t^r f(t, y) e^{-st - \frac{(x-y)^2}{4}} dy dt$$

On differentiating both sides w. r. t. s we get,

$$(-1)^{r+1} F^{(r+1)}(s, x) \\ = \frac{d}{ds} \left\{ \frac{1}{\sqrt{4\pi}} \int_0^\infty \int_0^\infty e^{-st - \frac{(x-y)^2}{4}} t^r f(t, y) dy dt \right\}$$

$$(-1)^{r+1} F^{(r+1)}(s, x) \\ = \frac{1}{\sqrt{4\pi}} \int_0^\infty \int_0^\infty t^{r+1} f(t, y) e^{-st - \frac{(x-y)^2}{4}} dy dt$$

$$LW \{ t^{r+1} f(t, y) \} = (-1)^{r+1} \frac{d^{r+1}}{ds^{r+1}} F(s, x) \quad (4.4)$$

\Rightarrow Theorem is true for $n = r + 1$.

Hence by induction, the theorem is true for all positive integral values of n .

ii) Given that, $LW \{ f(t, y) \} = F(s, x)$

$$F(s, x) = LW \{ f(t, y) \} = \frac{1}{\sqrt{4\pi}} \int_0^\infty \int_0^\infty f(t, y) e^{-st - \frac{(x-y)^2}{4}} dy dt$$

Now,

$$F'(s, x) = \frac{d}{dx} F(s, x) = \frac{d}{dx} \left\{ \frac{1}{\sqrt{4\pi}} \int_0^\infty \int_0^\infty f(t, y) e^{-st - \frac{(x-y)^2}{4}} dy dt \right\} \quad (4.5)$$

$$= \frac{-1}{2} LW \{ (x-y) f(t, y) \}$$

$$LW \{ (x-y) f(t, y) \} = (-1)(2) \frac{d}{dx} F(s, x) \quad (4.6)$$

i.e. the theorem is true for $n = 1$.

Next, let this theorem be true for $n = r$, then we have

$$LW \{ (x-y)^r f(t, y) \} = (-1)^r (2)^r \frac{d^r}{dx^r} F(s, x)$$

$$(-1)^r (2)^r \frac{d^r}{dx^r} [F(s, x)] = LW \{ (x-y)^r f(t, y) \}$$

$$= \frac{1}{\sqrt{4\pi}} \int_0^\infty \int_0^\infty (x-y)^r f(t, y) e^{-st - \frac{(x-y)^2}{4}} dy dt$$

Now differentiating both sides w. r. t. x we get,

$$\begin{aligned}
 & (-1)^r (2)^r \frac{d^{r+1}}{dx^{r+1}} [F(s, x)] \\
 &= \frac{d}{dx} \left\{ \frac{1}{\sqrt{4\pi}} \int_0^\infty \int_0^\infty (x-y)^r f(t, y) e^{-st - \frac{(x-y)^2}{4}} dy dt \right\} \\
 &= \frac{-1}{2} LW \{ (x-y)^{r+1} f(t, y) \} \\
 & LW \{ (x-y)^{r+1} f(t, y) \} = (-1)^{r+1} (2)^{r+1} F^{r+1}(s, x) \\
 & \quad (4.7)
 \end{aligned}$$

\Rightarrow Theorem is true for $n = r + 1$.

Hence by induction, the theorem is true for all positive integral values of n .

5. Conclusion:

In this paper we have introduced LW transform of a function $f(t, y)$ and seen how it exists. Also we have proved another theorem which is useful to solve certain equation.

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